

Analytical calculation of the nucleation rate for first order phase transitions beyond the thin wall approximation

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Abstract. First order phase transitions in general proceed via nucleation of bubbles. A theoretical basis for the calculation of the nucleation rate is given by the homogeneous nucleation theory of Langer and its field theoretical version of Callan and Coleman. We have calculated the nucleation rate beyond the thin wall approximation by expanding the bubble solution and the fluctuation determinant in powers of the asymmetry parameter. The result is expressed in terms of physical model parameters.

1 Introduction

First order phase transitions are a common phenomenon in statistical mechanics and in field theory [1]. They are characterized by the discontinuous change of an order parameter or other physical quantities as some driving parameter, e.g. temperature, is varied. In general they are associated with a latent heat. In the theory of elementary particles different phase transitions, which play a role in the evolution of the early universe, are predicted to be of first order. Among them is the electroweak phase transition, which has been investigated intensively in recent years, and the grand unification phase transition, which might be related to the inflationary epoch of the universe [2].

There are different mechanisms by which a first order phase transition can take place. In many cases it proceeds via nucleation of bubbles. Consider evaporation for example. If the temperature crosses the transition point the system enters a metastable state. In this state bubbles of the new, stable phase form spontaneously, which may then expand and lead to the completion of the transition to the new phase. Such metastable states have first been mentioned by Fahrenheit [3]. A theory of the formation of bubbles in liquid systems has been developed by Becker and Döring [4]. In the framework of the Ginzburg-Landau theory of phase transitions a phenomenological treatment was given by Cahn and Hilliard [5]. The theory of bubble nucleation was put on a profound theoretical basis by Langer [6–8]. His approach allows for a systematic treatment of the nucleation rate. A review is given in [9]. In the context of quantum field theory the nucleation theory was developed by Voloshin et al. [10], Callan

and Coleman [11,12] and Affleck [13]. Callan and Coleman presented an approach to the decay of an unstable vacuum in the framework of Euclidean quantum field theory. Although developed independently, it is very similar to Langer's formulation in terms of functional integrals. A nice exposition is given in Coleman's book [14].

The starting point of nucleation theory is the classical Ginzburg-Landau potential for the order parameter. It has an absolute minimum, corresponding to the stable phase, and one (or more) other minima. When the first order phase transition is approached, a previously higher minimum gets lower and lower, and when the transition point is crossed, it becomes the new absolute minimum. There is a barrier between the minima such that the system does not immediately go over into the new minimum but remains in a metastable state. This state is stable against small fluctuations. Due to fluctuations small regions (bubbles) of the stable phase may form spontaneously. Their creation leads to a gain in energy proportional to the volume,

$$-\mathcal{H}_V = \frac{4\pi}{3} R^3 \eta, \quad (1)$$

where R is the radius of the bubble and η is the difference of the potentials between the two minima. On the other hand a surface energy

$$\mathcal{H}_S = 4\pi R^2 \sigma, \quad (2)$$

where σ is the surface tension, has to be supplied, and the total energy associated with the bubble is approximately given by

$$\mathcal{H}_b(R) = \mathcal{H}_S + \mathcal{H}_V = 4\pi R^2 \left(\sigma - \frac{R}{3} \eta \right). \quad (3)$$

For small R this function increases with R so that small bubbles tend to shrink back to zero. Only if the radius

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exceeds the critical size of

$$R_c = \frac{2\sigma}{\eta}, \quad (4)$$

where

$$\frac{d\mathcal{H}_b(R_c)}{dR} = 0, \quad (5)$$

the bubble will expand and lead to the transformation of the metastable phase into the stable one in the whole volume.

The process described above is called homogeneous nucleation in contrast to heterogeneous nucleation, where impurities, like dust or ice crystals, trigger the phase transition. The cosmological phase transitions mentioned earlier are homogeneous. The formation of bubbles in homogeneous nucleation theory is analogous to quantum mechanical tunneling through a potential barrier. In fact, the description of tunneling by means of Euclidean functional integrals leads to a completely equivalent formalism. This fact is the basis of the relation between Langer's work and that of Callan and Coleman.

The rate in which the phase transition proceeds is essentially determined by the average time until a critical bubble forms spontaneously. A critical bubble of radius R_c is a solution of the field equations coming from the Ginzburg-Landau Hamiltonian. It has an energy $\mathcal{H}_c = \mathcal{H}_b(R_c)$. The nucleation rate Γ per time and volume is proportional to the Boltzmann factor of a critical bubble and can be written as

$$\Gamma = \mathcal{A}e^{-\mathcal{H}_c}, \quad (6)$$

which has already been found by Arrhenius [15].

For practical applications it is important to know the prefactor \mathcal{A} . Langer's theory gives an expression for \mathcal{A} in terms of the determinant of the operator of fluctuations around the critical bubble. It is the main object of this article to calculate the nucleation rate including the prefactor in the framework of scalar field theory, i.e., Ginzburg-Landau theory with fluctuations. Some elements of the calculation have been supplied by Langer [6], but a complete analytical calculation has been missing in the literature so far. A numerical method for the evaluation of the nucleation rate has been presented by Baacke and Kiselev [16].

In general it is not possible to find an analytical solution of the field equations for finite potential differences η . An approximation, where the field equations can be solved exactly, is the "thin wall approximation" [17]. This is the limiting case where η is much smaller than the height of the potential barrier. In this case the thickness of the wall of a critical bubble is much smaller than its radius, and its density profile can be approximated by a step function.

In this article we calculate the nucleation rate analytically beyond the thin wall approximation. We do this by expanding all quantities in powers of η and calculating the logarithm of the functional determinant of the fluctuation operator in terms of powers of η . The leading term corresponds to the thin wall approximation. For the calculation of the determinant we employ the Seeley expansion of the

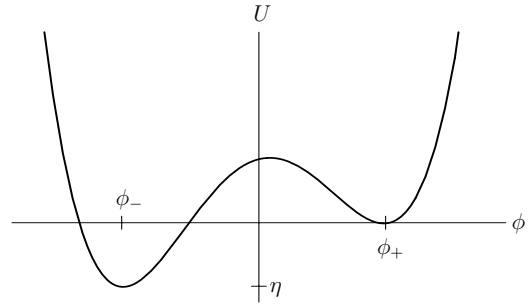


Fig. 1. The potential U with false (ϕ_+) and true vacuum (ϕ_-)

associated heat kernel on the one hand, and the spectrum of the fluctuation operator on the other hand. Ultraviolet divergencies require renormalization as usual. The resulting expression for the nucleation rate Γ is expressed in terms of renormalized parameters of the effective potential.

2 Nucleation theory

As shown by Langer [6,8], the nucleation rate Γ , that is the decay probability per time and per volume of a metastable state represented by a local minimum of a potential, is proportional to the imaginary part of a certain energy:

$$\Gamma = -2 \operatorname{Im} E. \quad (7)$$

The energy E is given by the logarithm of a functional integral

$$\mathcal{N} \int [d\phi] e^{-\mathcal{H}(\phi)}, \quad (8)$$

with appropriate boundary conditions, where $\phi(x)$ is the local order parameter and \mathcal{H} is the Ginzburg-Landau Hamiltonian (a factor $k_B T$ has been absorbed into \mathcal{H}). The Hamiltonian is given by

$$\mathcal{H}(\phi) = \int d^3x \left[\frac{1}{2} (\partial_\mu \phi(x))^2 + U(\phi(x)) \right], \quad (9)$$

with an asymmetric potential U . We consider a potential of the type depicted in Fig. 1, with a metastable minimum at $\phi = \phi_+$ and a stable minimum at $\phi = \phi_-$. Following Coleman we call the phase corresponding to the minimum at ϕ_+ the false vacuum and the one corresponding to the minimum at ϕ_- the true vacuum.

In a semiclassical approach the desired imaginary part of the functional integral can be obtained by means of a saddle point expansion around the classical solution which corresponds to the transition from the false to the true vacuum [6,11,12]. This solution describes a critical bubble. A bubble centered at the origin is represented by a radial symmetric function $\phi_c(r)$, which depends on $r = \sqrt{x_\mu x_\mu}$ only. The boundary condition at infinity

$$\lim_{r \rightarrow \infty} \phi_c(r) = \phi_+ \quad (10)$$

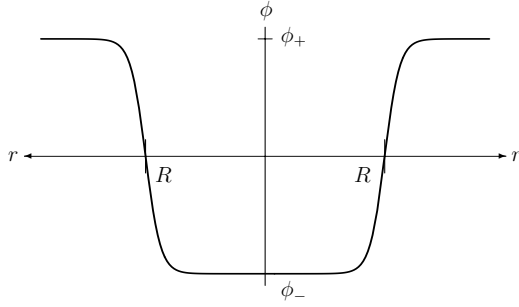


Fig. 2. Profile of the critical bubble

reflects that there is false vacuum outside the bubble. The presence of the true vacuum inside the bubble means that the value of the field at the center is near ϕ_- :

$$\phi_c(0) \approx \phi_- . \quad (11)$$

Due to differentiability of $\phi(x)$ we must have

$$\left. \frac{d\phi_c}{dr} \right|_{r=0} = 0 . \quad (12)$$

The field equation for the bubble solution is

$$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = U'(\phi) . \quad (13)$$

If we interpret r as time and ϕ as the coordinate of a particle, then this equation equals the equation of motion for a point particle in the reversed potential $-U$ with a time-dependent friction term. From the form of the potential it is intuitively clear that there is a unique value of $\phi_c(0)$ near ϕ_- , where the particle starts with zero velocity, then rolls down the slope and climbs up the other hill to approach its top ϕ_+ asymptotically as time goes to infinity. In fact, Coleman, Glaser and Martin [18] have proved that such a radial-symmetric non-trivial solution exists and that it is the one with smallest energy apart from the trivial solution $\phi \equiv \phi_+$. The qualitative form of the solution is as in Fig. 2, which shows a cross-section through the critical bubble.

For fields $\phi(x)$ near the classical solution ϕ_c the Hamiltonian can be expanded up to quadratic terms as

$$\mathcal{H}[\phi] = \mathcal{H}_c + \frac{1}{2} \int d^3x (\phi(x) - \phi_c(x)) \mathcal{M} (\phi(x) - \phi_c(x)) + \dots , \quad (14)$$

with the energy of the critical bubble,

$$\mathcal{H}_c = \mathcal{H}[\phi_c] , \quad (15)$$

and the fluctuation operator

$$\mathcal{M} = -\partial^2 + U''(\phi_c(x)) . \quad (16)$$

Due to translation invariance the operator \mathcal{M} has three zero modes proportional to $\partial_\mu \phi_c(x)$. Furthermore, there is one negative mode, which is related to the metastability

of the false vacuum. Namely, as has been indicated in the introduction, expansion or contraction of the critical bubble lowers its energy, which means that the corresponding mode belongs to a negative eigenvalue of \mathcal{M} . A proof of the existence of a single negative mode under rather general assumptions has been given by Coleman [19].

From the work of Langer and of Callan and Coleman it follows that the functional integral under consideration acquires an imaginary part, which is proportional to

$$|\lambda_- \det' \mathcal{M}|^{-1/2} e^{-\mathcal{H}_c} \quad (17)$$

in the Gaussian approximation, where \det' is the determinant without negative and zero modes, and λ_- is the negative eigenvalue of \mathcal{M} . If multibubble solutions are taken into account in a dilute gas approximation, the final result for the nucleation rate is obtained as

$$\Gamma = \left(\frac{\mathcal{H}_c}{2\pi} \right)^{3/2} \frac{1}{\sqrt{|\lambda_-|}} \left| \frac{\det' \mathcal{M}}{\det \mathcal{M}^{(0)}} \right|^{-1/2} e^{-\mathcal{H}_c} . \quad (18)$$

Here the operator $\mathcal{M}^{(0)}$ is the Helmholtz operator defined by

$$\mathcal{M}^{(0)} = -\partial^2 + U''(\phi_+) . \quad (19)$$

The expression (18) is of the form announced in the introduction. In this work the above expression, in particular the functional determinant, will be evaluated by field theoretic methods.

3 The bubble solution

We consider the standard Ginzburg-Landau potential consisting of a symmetric double-well term,

$$U_s = \frac{g}{4!} (\phi^2 - v^2)^2 , \quad (20)$$

and an additional asymmetric term:

$$U = U_s + \frac{\eta}{2v} (\phi - v) + U_0 . \quad (21)$$

The constant

$$U_0 = \frac{3\eta^2}{8v^4g} + \frac{9\eta^3}{16v^8g^2} + O(\eta^4) \quad (22)$$

is chosen such that $U(\phi_+) = 0$ (cf. Fig. 1). The parameter η fixes the asymmetry of the potential. In particular, the difference between the values of the potential at its minima ϕ_\pm is

$$U(\phi_+) - U(\phi_-) = 2\eta + O(\eta^3) . \quad (23)$$

We define a bare mass parameter m in terms of the symmetric part U_s :

$$m^2 = \left. \frac{\partial^2}{\partial \phi^2} U_s(\phi) \right|_{\phi=v} = \frac{gv^2}{3} . \quad (24)$$

The field equation for radially symmetric fields is

$$-\frac{d^2\phi}{dr^2} - \frac{2}{r} \frac{d\phi}{dr} + \frac{g}{6}\phi(\phi^2 - v^2) + \frac{\eta}{2v} = 0. \quad (25)$$

Before we turn to a systematic approach to solving this equation let us consider the thin wall approximation. Inspection of the field equation in the light of the mechanical analogue mentioned in the introduction shows that for small η the solution is nearly equal to $-v$ inside a sphere of radius R and nearly equal to $+v$ outside. The region where ϕ differs significantly from these values is a thin shell of thickness θ . The thin wall approximation amounts to

$$\phi = \begin{cases} -v, & \text{for } r < R - \theta/2 \\ \phi_k, & \text{for } R - \theta/2 < r < R + \theta/2 \\ +v, & \text{for } r > R + \theta/2, \end{cases} \quad (26)$$

where the ‘‘kink’’

$$\phi_k(r) = v \tanh\left(\frac{m}{2}(r - R)\right) \quad (27)$$

is a solution of

$$-\frac{d^2\phi}{dr^2} + \frac{g}{6}\phi(\phi^2 - v^2) = 0 \quad (28)$$

and $\theta = 4/m$.

The energy of a bubble solution can be written as

$$\begin{aligned} \mathcal{H}_c &= 4\pi \int_0^\infty dr r^2 \left[\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + U_s(\phi) + U_0 \right] \\ &+ \frac{2\pi\eta}{v} \int_0^\infty dr r^2 (\phi - v). \end{aligned} \quad (29)$$

The second term is substantially different from zero only inside the bubble. It yields the volume contribution

$$\mathcal{H}_V = -\frac{4\pi}{3} R^3 \eta. \quad (30)$$

The first term gets a substantial contribution only inside the wall,

$$\begin{aligned} \mathcal{H}_S &= 4\pi \int_{R-\theta/2}^{R+\theta/2} dr r^2 \left[\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + U_s(\phi) + U_0 \right] \\ &\approx 4\pi R^2 \int dr \left(\frac{d\phi}{dr} \right)^2 = 4\pi R^2 \sigma, \end{aligned} \quad (31)$$

with

$$\sigma = 2 \frac{m^3}{g}. \quad (32)$$

For small asymmetries η the critical radius $R_c = 2\sigma/\eta$, for which the energy is stationary, gets large. Therefore the wall is indeed thin compared to the size of the bubble. The total energy in this approximation is

$$\mathcal{H}_c = \frac{16\pi\sigma^3}{3\eta^2}. \quad (33)$$

For finite η the solution of (25) cannot be written in closed form. In our approach the solution is constructed by means of an expansion in powers of η . It is convenient to introduce dimensionless variables

$$\tilde{r} = \frac{m}{2}r, \quad \tilde{R} = \frac{m}{2}R, \quad \xi = \tilde{r} - \tilde{R}, \quad \tilde{\eta} = \frac{g}{2m^4}\eta, \quad (34)$$

$$\varphi(\xi) = \frac{1}{v}\phi(r), \quad (35)$$

where the value of \tilde{R} will be fixed later. The field equation in these variables is

$$-\frac{d^2\varphi}{d\xi^2} - \frac{2}{\xi + \tilde{R}} \frac{d\varphi}{d\xi} + 2\varphi(\varphi^2 - 1) + \frac{4}{3}\tilde{\eta} = 0. \quad (36)$$

Based on the thin wall approximation we write a Laurent series as an ansatz for the critical radius,

$$\tilde{R} = \frac{a_{-1}}{\tilde{\eta}} + a_0 + a_1\tilde{\eta} + a_2\tilde{\eta}^2 + \dots, \quad (37)$$

and expand the field equation into powers of $\tilde{\eta}$:

$$\begin{aligned} -\frac{d^2\varphi}{d\xi^2} - \frac{2}{a_{-1}} \tilde{\eta} \frac{d\varphi}{d\xi} + \tilde{\eta}^2 \frac{2(\xi + a_0)}{a_{-1}^2} \frac{d\varphi}{d\xi} \\ + 2\varphi(\varphi^2 - 1) + \frac{4}{3}\tilde{\eta} + O(\tilde{\eta}^3) = 0. \end{aligned} \quad (38)$$

Its solution is obtained perturbatively up to second order by means of the expansion

$$\varphi = \varphi_0 + \tilde{\eta}\varphi_1 + \tilde{\eta}^2\varphi_2 + O(\tilde{\eta}^3). \quad (39)$$

To zeroth order we get the well-known kink,

$$\varphi_0(\xi) = \tanh(\xi). \quad (40)$$

The field equation to first order fixes the leading coefficient in \tilde{R} , (37), as

$$a_{-1} = 1. \quad (41)$$

The first order solution obeying the correct boundary condition at $\xi \rightarrow \infty$ reads

$$\varphi_1 = -\frac{1}{3} - b \operatorname{sech}^2 \xi \quad (42)$$

with a free parameter b . The constant term reflects the shift of the minimum

$$\varphi_\pm = \pm 1 - \frac{\tilde{\eta}}{3} \mp \frac{\tilde{\eta}^2}{6} - \frac{4\tilde{\eta}^3}{27} + O(\tilde{\eta}^4). \quad (43)$$

The term proportional to b can be traded against a shift in the critical radius \tilde{R} in the lowest order solution according to

$$\begin{aligned} \tanh(\xi - b\tilde{\eta}) &= \tanh \xi - b\tilde{\eta} \operatorname{sech}^2 \xi - b^2 \tilde{\eta}^2 \operatorname{sech}^3 \xi \sinh \xi \\ &+ O(\tilde{\eta}^3). \end{aligned} \quad (44)$$

We can therefore set $b = 0$ and remain with

$$\varphi_1 = -\frac{1}{3}. \quad (45)$$

The equation to second order implies

$$a_0 = 0 \quad (46)$$

and has the solution

$$\begin{aligned} \varphi_2(\xi) = & -\frac{\xi}{2}(\tanh \xi - 1) + \frac{\xi}{6}(\cosh \xi - \sinh \xi)^2 \\ & -\frac{1}{2}\xi^2 \operatorname{sech}^2 \xi - \frac{7}{12}\xi \operatorname{sech}^2 \xi - \ln(1 + e^{-2\xi}) \\ & \times \left(\frac{1}{2}\xi \operatorname{sech}^2 \xi + \frac{1}{2}\tanh \xi \right) - \frac{1}{3}\ln(1 + e^{-2\xi}) \\ & \times \sinh \xi \cosh \xi - \frac{1}{12}\tanh \xi + \frac{1}{2}\operatorname{sech}^2 \xi T(\xi), \quad (47) \end{aligned}$$

where we define

$$T(\xi) = \int_0^\xi \xi' \tanh \xi' d\xi'. \quad (48)$$

Whereas the first order solution only corresponds to shifts of the minimum and of the critical radius, the second order solution describes true deformations of the bubble. The boundary condition at $r = 0$, i.e. $\xi = -\tilde{R}$, is fulfilled order by order in $\tilde{\eta}$. For example, the leading order solution yields

$$\varphi'_0(-\tilde{R}) = e^{-2/\tilde{\eta}}(4 + O(\tilde{\eta})), \quad (49)$$

which vanishes to all orders in $\tilde{\eta}$. Similar observations hold in higher orders.

With the expression for φ we can calculate the energy of a bubble, which in dimensionless quantities is given by

$$\begin{aligned} \mathcal{H}_c = & \frac{12\pi m}{g} \int_0^\infty d\tilde{r} \tilde{r}^2 \left\{ (\varphi'(\xi))^2 + [(\varphi^2(\xi) - 1)^2 \right. \\ & \left. - (\varphi_+^2 - 1)^2] + \left[\frac{8}{3}\tilde{\eta} (\varphi(\xi) - \varphi_+) \right] \right\}. \quad (50) \end{aligned}$$

The integrands are centered around the critical radius $\tilde{r} = \tilde{R}$. The integration range in ξ can be extended to the whole real axis. The error coming from this is proportional to factors of the type $e^{-\text{const.}/\tilde{\eta}}$ and vanishes to all orders in $\tilde{\eta}$.

From the parity of the functions $\varphi_k(\xi)$ it follows that the expression for the energy is of the form

$$\begin{aligned} \mathcal{H}_c = & \frac{12\pi m}{g} (O_0 \tilde{R}^2 + P_0) + \tilde{\eta} (L_1 \tilde{R} + v_1 \tilde{R}^3) \\ & + \tilde{\eta}^2 (O_2 \tilde{R}^2 + P_2) + \tilde{\eta}^3 (L_3 \tilde{R} + v_3 \tilde{R}^3) + O(\tilde{\eta}^4). \quad (51) \end{aligned}$$

An expression for the critical radius \tilde{R} is obtained from the condition

$$\frac{d\mathcal{H}_c}{d\tilde{R}} \stackrel{!}{=} 0. \quad (52)$$

Explicit calculation of the coefficients leads to two more terms in the Laurent series for \tilde{R} :

$$\tilde{R} = \frac{1}{\tilde{\eta}} + 0 + \frac{2 - 3\pi^2}{36}\tilde{\eta} + 0 \cdot \tilde{\eta}^2 + O(\tilde{\eta}^3), \quad (53)$$

so that the bubble is now completely determined to second order. For the energy we get

$$\mathcal{H}_c = \frac{12\pi m}{g} \left[\frac{8}{9} \frac{1}{\tilde{\eta}^2} + \frac{2(4 - 9\pi^2)}{81} + O(\tilde{\eta}^2) \right]. \quad (54)$$

4 The heat kernel of \mathcal{M}

Our main task is to calculate the determinant ratio

$$\frac{\det' \mathcal{M}}{\det \mathcal{M}^{(0)}}, \quad (55)$$

which is part of the prefactor in the nucleation rate. In dimensionless variables the corresponding operators are

$$M = \frac{4}{m^2} \mathcal{M} = -\frac{\partial^2}{\partial \xi_\mu \partial \xi_\mu} + 6\varphi^2(\xi) - 2 \quad (56)$$

and

$$M^{(0)} = \frac{4}{m^2} \mathcal{M}^{(0)} = -\frac{\partial^2}{\partial \xi_\mu \partial \xi_\mu} + 6\varphi_+^2 - 2. \quad (57)$$

Substituting the bubble solution yields for the potential

$$\begin{aligned} V(\xi) = & 6\varphi^2(\xi) - 2 \\ = & V_0(\xi) + \tilde{\eta} V_1(\xi) + \tilde{\eta}^2 V_2(\xi) + O(\tilde{\eta}^3), \quad (58) \end{aligned}$$

with the coefficients

$$V_0(\xi) = -6 \operatorname{sech}^2 \xi + 4, \quad (59)$$

$$V_1(\xi) = -4 \tanh \xi, \quad (60)$$

$$\begin{aligned} V_2(\xi) = & \frac{2}{3} + \xi [4 \tanh \xi + 4 \sinh \xi \cosh \xi - (7 + 6 \ln 2) \\ & \times \operatorname{sech}^3 \xi \sinh \xi] - \tanh \xi \ln(\cosh \xi) (6\xi \operatorname{sech}^2 \xi \\ & + 4 \cosh \xi \sinh \xi + 6 \tanh \xi) - (1 + 6 \ln 2) \\ & \times \tanh^2 \xi - 4 \ln 2 \sinh^2 \xi \\ & + 6 \operatorname{sech}^3 \xi \sinh \xi T(\xi). \quad (61) \end{aligned}$$

For the free operator we find accordingly

$$V^{(0)} = 6\varphi_+^2 - 2 = V_0^{(0)} + \tilde{\eta} V_1^{(0)} + \tilde{\eta}^2 V_2^{(0)} + O(\tilde{\eta}^3), \quad (62)$$

with

$$V_0^{(0)} = +4, \quad (63)$$

$$V_1^{(0)} = -4, \quad (64)$$

$$V_2^{(0)} = -\frac{4}{3}. \quad (65)$$

The scaling dimensions of \mathcal{M} and $\mathcal{M}^{(0)}$ are equal, as we have checked with the help of their zeta functions (see below). Therefore in the determinant ratio the four removed eigenvalues lead to a supplementary factor:

$$\frac{\det' \mathcal{M}}{\det \mathcal{M}^{(0)}} = \left(\frac{4}{m^2} \right)^4 \frac{\det' M}{\det M^{(0)}}. \quad (66)$$

To get the same number of eigenvalues in the numerator and denominator, we also remove the four lowest eigenvalues $\omega_0^{(0)} = V^{(0)}$ of the free operator $M^{(0)}$, which are equal to each other:

$$\frac{\det' M}{\det M^{(0)}} = (\omega_0^{(0)})^{-4} \frac{\det' M}{\det M^{(0)}}. \quad (67)$$

The prime indicates the omission of the four lowest eigenvalues. The remainder is written in Schwinger's proper time representation:

$$\ln \frac{\det' M}{\det' M^{(0)}} = - \int_0^\infty \frac{dt}{t} \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}). \quad (68)$$

In the integrand we recognize the heat kernels $\exp(-tM)$ of the operators M and $M^{(0)}$, respectively. The distribution of the large eigenvalues determines the behaviour of the heat kernel for small t , whereas the lowest eigenvalues determine the large- t behaviour. The strategy for the calculation of the integral over t is to divide the integration range into a small- t part, where the heat kernel is approximated by an asymptotic expansion, and a large- t part, where the spectrum is employed [20].

Let us consider the small- t region first. For small and positive t an asymptotic expansion for the heat kernels, the so-called Seeley expansion, exists [21]. For the trace of the heat kernels in D dimensions it is of the form

$$\text{Tr}(e^{-tM} - e^{-tM^{(0)}}) = (4\pi t)^{-D/2} \sum_{n=1}^{\infty} t^n \mathcal{O}_n. \quad (69)$$

There are various methods for the calculation of the coefficients \mathcal{O}_n . Our calculation is based on the insertion of a plane wave basis in the manner of [20, 22]. By means of partial integrations we managed to express the coefficients in terms of the potential V and the Laplacean ∂^2 , which for a radial symmetric potential like ours depends only on \tilde{r} [23]. Inserting the potential $V(\xi)$, (58), and substituting the expression for \tilde{R} , (53), we obtained with the help of Mathematica [24] the result

$$\begin{aligned} & \text{Tr}(e^{-tM} - e^{-tM^{(0)}}) \\ &= \frac{1}{(4\pi t)^{3/2}} \left[\frac{1}{\tilde{\eta}^2} \left(\frac{112\pi}{3} t - \frac{160\pi}{3} t^2 + \frac{832\pi}{15} t^3 \right. \right. \\ & \quad \left. \left. - \frac{11392\pi}{315} t^4 + \frac{3328\pi}{315} t^5 - \frac{49664\pi}{24255} t^6 + \dots \right) \right. \\ & \quad \left. + (-39.4801 t + 372.46 t^2 - 541.384 t^3 + 658.823 t^4 \right. \\ & \quad \left. - 913.886 t^5 + 1225.92 t^6 - \dots) + O(\tilde{\eta}^2) \right]. \quad (70) \end{aligned}$$

From the leading term $t^{-1/2}$ it can be seen that the integral over t diverges at small t . It is well known that this divergence is another disguise of the usual ultraviolet divergencies of quantum field theory. They can be treated by means of dimensional regularization. It is a peculiarity of the three-dimensional case that the regularized expression is identical to the zeta-function regularized one without any additional finite contribution [25]. The zeta function of an operator A is in general defined by

$$\zeta_A(z) = \text{Tr} A^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \text{Tr}(e^{-At}) \quad (71)$$

for sufficiently large $\text{Re } z$, where the integral converges, and continued analytically to the rest of the complex

plane. The zeta-function regularized determinant is then given by

$$\ln \det A = - \left. \frac{d}{dz} \zeta_A(z) \right|_{z=0}. \quad (72)$$

As we are interested in the ratio of two determinants, and want to exclude zero and negative modes, we define

$$\begin{aligned} \zeta'(z) &= \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}) \quad \text{for} \\ & \text{Re } z > 1. \end{aligned} \quad (73)$$

It can be continued analytically to $z = 0$ by separating the first term in the Seeley expansion of $\text{Tr}(e^{-tM} - e^{-tM^{(0)}})$, Eqs. (69),(70):

$$\begin{aligned} \zeta'(z) &= \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \left\{ \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}) - \Theta(1-t) \right. \\ & \quad \left. \times \frac{1}{(4\pi t)^{3/2}} t \mathcal{O}_1 \right\} + \frac{1}{(4\pi)^{3/2}} \frac{1}{\Gamma(z)} \frac{\mathcal{O}_1}{z-1/2}. \end{aligned} \quad (74)$$

Now the ratio of determinants can be expressed as

$$\ln \frac{\det' M}{\det' M^{(0)}} = - \left. \frac{d}{dz} \zeta'(z) \right|_{z=0}. \quad (75)$$

At this point let us check the relative dimensions of the two operators. If the dimensions of M and $M^{(0)}$ differ by some number u this means that

$$\frac{\det' \lambda M}{\det' \lambda M^{(0)}} = \lambda^u \frac{\det' M}{\det' M^{(0)}}. \quad (76)$$

Representing the ratio of determinants by the derivative of the zeta-function one obtains

$$u = \zeta'(0). \quad (77)$$

In the expression (74) for the analytically continued zeta function the integral is convergent. Since $1/\Gamma(z)$ vanishes at $z = 0$, we find

$$\zeta'(0) = 0, \quad (78)$$

so that the operators have equal dimensions as announced above.

5 Calculation of the determinant

As mentioned above, the strategy is to split the t -integration into a small- t part and a large- t part. Therefore we introduce a parameter Λ separating the two regions, and split the zeta function as

$$\zeta'(z) = \zeta'_<(z) + \zeta'_>(z) \quad (79)$$

with

$$\zeta'_<(z) = \frac{1}{\Gamma(z)} \int_0^\Lambda dt t^{z-1} \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}) \quad (80)$$

and $\zeta'_>(z)$ correspondingly. In the same way the log of the ratio of determinants is split as

$$I \equiv \ln \frac{\det' M}{\det' M^{(0)}} = I_< + I_> \quad (81)$$

with

$$I_< = - \left. \frac{d}{dz} \zeta'_<(z) \right|_{z=0}, \quad (82)$$

$$I_> = - \left. \frac{d}{dz} \zeta'_>(z) \right|_{z=0}. \quad (83)$$

Because the t -integration is not singular for $t > \Lambda$, we can express $I_>$ directly as

$$I_> = - \int_{\Lambda}^{\infty} \frac{dt}{t} \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}). \quad (84)$$

5.1 High-frequency part

The behaviour of the heat kernel at small t is governed by the high frequencies in the spectrum of the operators. Therefore the contribution to the determinant from the integration over small t is its high-frequency part. The small- t expansion of the heat kernel is given in (70). For the calculation of $\zeta'_<(z)$ we have to subtract the contributions of the negative mode $\omega_- < 0$ and the three zero-modes of M , and to add the contribution of the four lowest eigenvalues of $M^{(0)}$:

$$\zeta'_<(z) = \frac{1}{\Gamma(z)} \int_0^{\Lambda} dt t^{z-1} \left\{ \frac{1}{(4\pi t)^{3/2}} \sum_{n=1}^{\infty} \mathcal{O}_n t^n - e^{t|\omega_-|} - 3 + 4e^{-t\omega_0^{(0)}} \right\}. \quad (85)$$

For $\text{Re } z > 1$ the integral can be performed:

$$\begin{aligned} \zeta'_<(z) &= \frac{1}{\Gamma(z)} \frac{1}{(4\pi)^{3/2}} \sum_{n=1}^{\infty} \frac{\mathcal{O}_n}{z - 3/2 + n} \Lambda^{z-3/2+n} \\ &\quad - \frac{|\omega_-|^{-z}}{\Gamma(z)} \int_0^{|\omega_-|\Lambda} ds s^{z-1} (e^s - 1) - 4 \frac{\Lambda^z}{\Gamma(z+1)} \\ &\quad + 4 \frac{(\omega_0^{(0)})^{-z}}{\Gamma(z)} \gamma(z, \omega_0^{(0)} \Lambda), \end{aligned} \quad (86)$$

where

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt \quad (87)$$

is an incomplete gamma function [26, sec. 6.5]. This expression can be continued analytically to $z = 0$, and the derivative at this point yields

$$\begin{aligned} I_< &= - \frac{1}{(4\pi)^{3/2}} \sum_{n=1}^{\infty} \left\{ \frac{\mathcal{O}_n}{n - 3/2} \Lambda^{n-3/2} \right\} \\ &\quad + \text{Ei}(|\omega_-|\Lambda) - \ln(|\omega_-|\Lambda) + 3\gamma + 4\text{E}_1(\omega_0^{(0)}\Lambda) \\ &\quad + 4 \ln(\omega_0^{(0)}\Lambda), \end{aligned} \quad (88)$$

where $\gamma = -\Gamma'(1) = 0.57721\dots$ is Euler's constant and

$$\text{Ei}(x) = -\text{P} \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \quad x > 0 \quad (89)$$

$$\text{E}_1(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt, \quad x > 0 \quad (90)$$

are the exponential integrals [26, sec. 5.1]. Here the explicit coefficients \mathcal{O}_n from (70) and the expressions for ω_- and $\omega_0^{(0)}$ given in the next section are to be inserted.

5.2 Low-frequency part

The low eigenvalues of M and $M^{(0)}$ determine the behaviour of the heat kernels for large t . We calculate the complete spectrum with the help of a perturbative expansion in $\tilde{\eta}$.

In spherical coordinates the eigenvalue equation

$$Mv(\tilde{r}) = \omega v(\tilde{r}), \quad (91)$$

goes by means of the usual transformation

$$v(\tilde{r}) = \frac{1}{\tilde{r}} \hat{\psi}(\tilde{r}) \quad (92)$$

and the shift

$$\xi = \tilde{r} - \tilde{R}, \quad \psi(\xi) = \hat{\psi}(\tilde{r}) \quad (93)$$

over into

$$\left[-\frac{d^2}{d\xi^2} + \frac{l(l+1)}{(\xi + \tilde{R})^2} + V(\xi) \right] \psi_{nl}(\xi) = \omega_{nl} \psi_{nl}(\xi). \quad (94)$$

where $V(\xi)$ is given in (58), and $l = 0, 1, 2, \dots$ is the angular quantum number. With the help of (53) the left hand side is expanded in powers of $\tilde{\eta}$. To lowest order one finds the Pöschl-Teller potential $V_0 = -6 \text{sech}^2 \xi + 4$, whose eigenvalues are known exactly [27]. There exist two discrete values

$$\omega_0^0 = 0, \quad \psi_0^0(\xi) = \sqrt{\frac{3}{4}} \text{sech}^2 \xi, \quad (95)$$

$$\omega_3^0 = 3, \quad \psi_3^0(\xi) = \sqrt{\frac{3}{2}} \sinh \xi \text{sech}^2 \xi, \quad (96)$$

and a continuum

$$\omega_k^0 = k^2 + 4, \quad k \in \mathbf{R}, \quad (97)$$

with the corresponding eigenfunctions

$$\psi_k^0(\xi) \sim e^{ik\xi} (3 \tanh^2 \xi - 1 - k^2 - 3ik \tanh \xi). \quad (98)$$

Because of the radial symmetry, the problem is here only defined along the half-axis $\tilde{r} > 0$. The boundary condition at $\tilde{r} = 0$, i.e. $\xi = -\tilde{R}$, is obeyed up to terms which vanish to all orders in $\tilde{\eta}$.

To first order in $\tilde{\eta}$ the corrections to the eigenvalues vanish.

For the second order, we use the following trick [28]. For every eigenfunction ψ_n^0 in the zeroth order there is an operator Ω_n , fulfilling the relation

$$[\Omega_n, (-\partial^2 + V_0)] \psi_n^0 = V_1 \psi_n^0. \quad (99)$$

The second correction in the eigenvalue is then given by

$$\omega_n^2 = \langle \psi_n^0 | V_1 \Omega_n + V_2 | \psi_n^0 \rangle + l(l+1). \quad (100)$$

This way, we find the second order for the discrete eigenvalues:

$$\omega_0^2 = l(l+1) - 2, \quad (101)$$

$$\omega_3^2 = l(l+1) + 3 - \pi^2, \quad (102)$$

using

$$\Omega_0 = \xi, \quad \Omega_3 = 2\xi - \frac{\cosh \xi}{\sinh \xi}. \quad (103)$$

So the discrete eigenvalues of M are

$$\omega_{0l} = \tilde{\eta}^2 (l(l+1) - 2) + O(\tilde{\eta}^4) \quad (104)$$

$$\omega_{3l} = 3 + \tilde{\eta}^2 (l(l+1) + 3 - \pi^2) + O(\tilde{\eta}^4). \quad (105)$$

They are $(2l+1)$ -fold degenerate.

In particular, the negative mode is given by

$$\omega_- = \omega_{00} = -2\tilde{\eta}^2 + O(\tilde{\eta}^4), \quad (106)$$

and the three zero modes are ω_{01} .

The band of eigenvalues ω_{0l} near zero gives a contribution to $I_{>}$ which reads

$$I_{>}^0(\Lambda) = - \int_{\Lambda}^{\infty} \frac{dt}{t} \sum_{l=2}^{\infty} (2l+1) e^{-\omega_{0l} t}. \quad (107)$$

This expression can be evaluated with the help of various nontrivial relations involving special functions. The result is

$$I_{>}^0(\Lambda) = -\frac{1}{\tilde{\eta}^2 \Lambda} - \frac{5}{3} \ln(\tilde{\eta}^2 \Lambda) + c_0 + O(\tilde{\eta}^2 \Lambda) \quad (108)$$

with

$$c_0 = \frac{9}{2} - \ln 54 - 4\zeta'_R(-1) - \frac{5}{3}\gamma = 0.21068\dots, \quad (109)$$

where $\zeta_R(z)$ is Riemann's zeta function. However, the corrections to the eigenvalues from next order perturbation theory would produce corrections of order $(\tilde{\eta}^2 \Lambda)^0$, and therefore we write

$$I_{>}^0(\Lambda) = -\frac{1}{\tilde{\eta}^2 \Lambda} - \frac{5}{3} \ln(\tilde{\eta}^2 \Lambda) + O((\tilde{\eta}^2 \Lambda)^0). \quad (110)$$

In a similar way the band of eigenvalues ω_{3l} near three gives a contribution

$$I_{>}^3(\Lambda) = -\frac{3}{\tilde{\eta}^2} \Gamma(-1, 3\Lambda) + O(\tilde{\eta}^0). \quad (111)$$

with the incomplete gamma function [26, sec. 6.5])

$$\Gamma(a, x) = \int_x^{\infty} e^{-t} t^{a-1} dt. \quad (112)$$

The term of order $\tilde{\eta}^0$ has been calculated, but is not displayed, because $I_{>}^0$ already contains an uncertainty of this order.

Now we turn to the remaining spectrum. The continuous eigenvalues of M and all eigenvalues of $M^{(0)}$ can be written in the form

$$\omega_{kl} = k^2 + V^{(0)} + \tilde{\eta}^2 l(l+1) + O(\tilde{\eta}^4), \quad k \geq 0, \quad (113)$$

with

$$V^{(0)} = 4 - 4\tilde{\eta} - \frac{4}{3}\tilde{\eta}^2 + O(\tilde{\eta}^3). \quad (114)$$

In order to calculate their contribution to the heat kernel one needs the difference of the spectral densities $\varrho_l(k)$ and $\varrho_l^{(0)}(k)$. We have calculated the spectral densities in the framework of perturbation theory. This was done by extracting the phase shifts $\delta_l(k)$ from the asymptotic behaviour of the wavefunctions and using the relation

$$\varrho_l(k) - \varrho_l^{(0)}(k) = \frac{1}{\pi} \frac{\partial \delta_l}{\partial k}. \quad (115)$$

It turns out that in n -th order of perturbation theory there are terms proportional to $(\tilde{\eta} \tilde{R})^n$, which contribute to the lowest order in $\tilde{\eta}$, because $\tilde{R} \sim 1/\tilde{\eta}$. Summing up all these terms we have been able to obtain the spectral densities to lowest order only. Omitting the details, the contribution of the continuous spectra to the trace of the heat kernels (leaving out the four lowest eigenvalues of $M^{(0)}$) is given by

$$\begin{aligned} & \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dk \left(\varrho_l(k) - \varrho_l^{(0)}(k) \right) e^{-\omega_{kl} t} + 4e^{-t\omega^{(0)}} \\ &= -\frac{1}{\tilde{\eta}^2 t} \left[e^{-3t} + 1 - \left(e^{-3t} \Phi(\sqrt{t}) + \Phi(\sqrt{4t}) \right) \right. \\ & \quad \left. - \frac{2}{3} V_1^{(0)} \sqrt{\frac{t}{4\pi}} e^{-4t} \right] + O(\tilde{\eta}^0), \end{aligned} \quad (116)$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (117)$$

is the error integral. From this we get the contribution to $I_{>}$ by integration over t :

$$\begin{aligned} I_{>}^k(\Lambda) &= \frac{1}{\tilde{\eta}^2} \left[3\Gamma(-1, 3\Lambda) - \frac{4}{3\sqrt{\pi}} \Gamma(-1/2, 4\Lambda) + \frac{1}{\Lambda\sqrt{\pi}} \right. \\ & \quad \left. \times \Gamma(1/2, \Lambda) - \int_{\Lambda}^{\infty} \frac{dt}{t^2} \Phi(\sqrt{t}) e^{-3t} \right] + O(\tilde{\eta}^0). \end{aligned} \quad (118)$$

We now have all contributions to $I_{>} = I_{>}^0 + I_{>}^3 + I_{>}^k$ available. In Fig. 4 they are presented as a function of Λ for $\tilde{\eta} = 0.1$. One can see that for $\Lambda > 1$ the contribution $I_{>}^0$ from the band near zero dominates the sum.

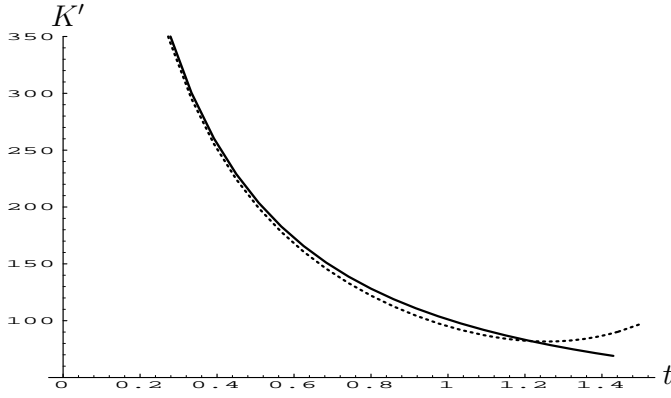


Fig. 3. The heat kernel K' as a function of t for $\tilde{\eta} = 0.1$. The dotted curve represents the Seeley expansion and the full curve is the approximation from summing over the eigenvalues

5.3 Composition of the determinant

With all pieces at hand the logarithm of the determinant

$$I = \ln \frac{\det' M}{\det' M^{(0)}} = I_{<}(\Lambda) + I_{>}(\Lambda) \quad (119)$$

can now be composed. The uncertainty in our result for $I_{<}$ is of order $\tilde{\eta}^2$. On the other hand, for $I_{>}$ there are already unknown contributions of order $\tilde{\eta}^0$. In the numerical evaluations we nevertheless include the constant terms from our calculation of $I_{>}^0$ and $I_{>}^3$. Since the result for I is strongly dominated by $I_{<}$, as will be seen below, the influence of the unknown corrections is expected to be numerically small.

In an exact calculation I would be independent of the artificial cutoff parameter Λ . Using the approximative expressions above, a dependence on Λ of course shows up, and we have to make a good choice. As a first guide we consider the heat kernel calculated from the Seeley expansion, (69), and from the eigenvalues, respectively. In Fig. 3 the kernel

$$K'(t) = \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}) \quad (120)$$

is shown as function of t for $\tilde{\eta} = 0.1$ from the two approximations.

The small- t approximation and the large- t approximation are in good agreement for $t < 1.3$. A value of Λ near 1.2, where the two curves intersect each other, appears to be reasonable for this value of $\tilde{\eta}$.

To make things more quantitative, the various contributions to I and the total sum are shown as a function of Λ in Fig. 4. A broad plateau, where the Λ -dependence is rather small can clearly be recognized.

An optimal value for Λ can be determined in the following way. As discussed above, the expression for I can be expanded in powers of $\tilde{\eta}$. The coefficients are functions of Λ . Requiring

$$\frac{dI}{d\Lambda} = 0 \quad (121)$$

order by order in $\tilde{\eta}$ leads to

$$\Lambda = 1.11672 + 17.3175 \tilde{\eta}^2 + O(\tilde{\eta}^3). \quad (122)$$

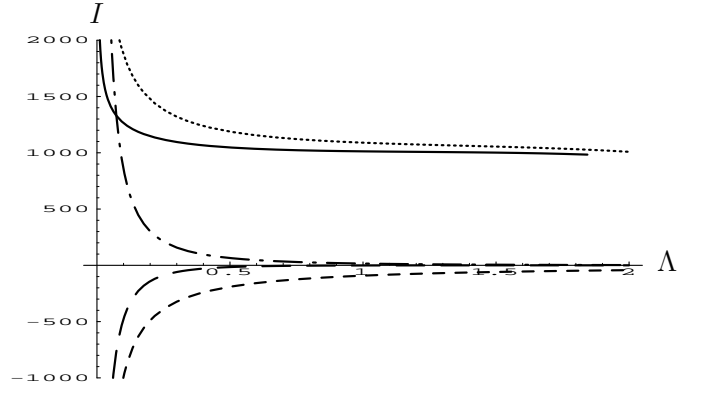


Fig. 4. The different contributions to I as a function of Λ for $\tilde{\eta} = 0.1$: $I_{<}^0$ (\cdots), $I_{>}^0$ ($---$), $I_{>}^3$ ($---$), $I_{>}^k$ ($- \cdot - \cdot$). The full curve is the sum of the four contributions

With this value one obtains

$$I(\tilde{\eta}) = \frac{10.158}{\tilde{\eta}^2} - \frac{5}{3} \ln \tilde{\eta}^2 - 13.63 + O(\tilde{\eta}^0) \quad (123)$$

with an error estimate of

$$\Delta I(\tilde{\eta}) = \frac{0.0527}{\tilde{\eta}^2} - 6.37 + O(\tilde{\eta}^0). \quad (124)$$

The systematic uncertainties of order $\tilde{\eta}^0$ come from the corresponding unknown terms in $I_{>}$. Their contribution should be numerically small, as can be seen from Fig. 4. The error ΔI is estimated from the error of the Seeley expansion, which in turn is taken to be given by its highest term.

5.4 Improved calculation of the determinant

The result for the determinant can be improved in various ways. First of all, one observes that the high frequency (small t) part dominates the result. Also, the low frequency part contains uncertainties of constant order in $\tilde{\eta}$. Therefore it is desirable to use as much information as possible from the high frequency part, i.e., from the Seeley expansion.

The determinant could be estimated from the Seeley expansion alone by introducing a smooth exponential cutoff [29]. With a free cutoff parameter μ one writes

$$\begin{aligned} K'(t) &= \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}) \\ &= e^{-\mu t} \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}) e^{\mu t}, \end{aligned} \quad (125)$$

and expands the modified kernel as

$$\begin{aligned} \text{Tr}'(e^{-tM} - e^{-tM^{(0)}}) e^{\mu t} \\ = \frac{1}{(4\pi t)^{3/2}} \sum_n h_n(\mu) t^n + \sum_n g_n(\mu) t^n, \end{aligned} \quad (126)$$

where the coefficients h_n come from the Seeley expansion (70) and the coefficients g_n from the subtracted negative

mode and zero modes. Due to the factor $\exp(-\mu t)$ the t -integration can be extended to infinity. In this way one gets an estimate for I from the small- t expansion alone.

We have used this method and obtained results which are in fair agreement with the earlier ones. It is, however, possible to introduce a further improvement. From the knowledge of the band of low-lying eigenvalues ω_{0l} it is possible to calculate its contribution to I completely [30]. The outcome is

$$I^0 = -\frac{5}{3} \ln \tilde{\eta}^2 + \frac{9}{2} - \ln 54 - 4\zeta'_R(-1) + O(\tilde{\eta}^2) \quad (127)$$

$$= -\frac{5}{3} \ln \tilde{\eta}^2 + 1.1727005 + O(\tilde{\eta}^2). \quad (128)$$

This result can be employed in the calculation of I by separating the contribution of this band of eigenvalues. This means that from the Seeley expansion above the small- t expansion of

$$\sum_{l=2}^{\infty} (2l+1)e^{-\omega_{0l}t} = \frac{1}{\tilde{\eta}^2 t} - \frac{5}{3} + O(\tilde{\eta}^2 t) \quad (129)$$

is subtracted and the rest is treated according to the exponential cutoff method. To the result the expression for I^0 is added finally.

The value of the cutoff-parameter μ has been obtained with the same procedure as in the case of Λ by requiring $dI/d\mu$ to vanish. For the log of the determinant we get in this way

$$I(\tilde{\eta}) = \frac{10.037}{\tilde{\eta}^2} - \frac{5}{3} \ln \tilde{\eta}^2 - 7.98 - 4\tilde{\eta} + O(\tilde{\eta}^2). \quad (130)$$

We have estimated the error by means of the Shanks extrapolation and error determination [31] and obtained

$$\Delta I(\tilde{\eta}) = \frac{0.00154}{\tilde{\eta}^2} - 0.084 - 0.795\tilde{\eta} + O(\tilde{\eta}^2). \quad (131)$$

A last improvement is based on the knowledge of the exact leading terms in the small- $\tilde{\eta}$ expansion of $I(\tilde{\eta})$ [30]. From the results for the discrete and continuous spectra of $M^{(0)}$ it is possible to derive the exact result

$$I = \frac{c}{\tilde{\eta}^2} - \frac{5}{3} \ln \tilde{\eta}^2 + O(\tilde{\eta}^0), \quad (132)$$

with

$$c = \frac{20}{3} + 3 \ln 3 = 9.9625. \quad (133)$$

So we write our final result for the determinants as

$$\begin{aligned} \ln \frac{\det' M}{\det M^{(0)}} &= I - 4 \ln \omega_0^{(0)} \\ &= \frac{c}{\tilde{\eta}^2} - \frac{5}{3} \ln \tilde{\eta}^2 - 13.52 + O(\tilde{\eta}^2). \end{aligned} \quad (134)$$

6 The nucleation rate

Having obtained the determinant of the fluctuation operator the nucleation rate can be calculated according to (18), where the energy of the critical bubble, (54), and the negative mode, (106), have to be inserted. In terms of the mass m and the dimensionless parameters $\tilde{\eta}$ and

$$u = \frac{g}{m} \quad (135)$$

the nucleation rate is

$$\begin{aligned} \Gamma &= \frac{m^3}{u^{3/2} \tilde{\eta}^{7/3}} \exp \left[- \left(\frac{32\pi}{3} \frac{1}{u} + \frac{c}{2} + O(u) \right) \frac{1}{\tilde{\eta}^2} \right. \\ &\quad \left. + \left(\frac{8\pi}{27} (9\pi^2 - 4) \frac{1}{u} + 6.845 + O(u) \right) + O(\tilde{\eta}^2) \right]. \end{aligned} \quad (136)$$

The parameters appearing in the Hamiltonian are, however, not immediately accessible in phenomenological applications or in a field theoretical context. More appropriate are the renormalised parameters, which are directly related to measurable quantities. Even more important in this context is the fact that in the dimensional regularisation scheme around $d = 3$ dimensions the divergencies in the relation between bare and renormalised parameters are not visible in the one-loop approximation. Therefore physical quantities should be expressed in terms of renormalised parameters.

We shall use the renormalised quantities as, e.g., specified in [25]. The renormalised mass m_R and the field renormalisation constant Z are defined in terms of the inverse propagator at small momenta:

$$G^{-1}(p) = \frac{1}{Z} \left(m_R^2 + p^2 + O(p^4) \right). \quad (137)$$

The renormalised mass is equal to the inverse of the second moment correlation length

$$\xi^{(2)} = \frac{1}{m_R}. \quad (138)$$

The renormalised field is given by

$$\phi_R = Z^{-1/2} \phi \quad (139)$$

and the renormalised field expectation value

$$v_R = Z^{-1/2} (v + \langle \phi \rangle) \quad (140)$$

correspondingly. The renormalised coupling, defined in terms of the mass and the field expectation value by

$$g_R = \frac{3m_R^2}{v_R^2}, \quad (141)$$

has dimensions of a mass. Its dimensionless counterpart is

$$u_R = \frac{g_R}{m_R}. \quad (142)$$

The renormalised dimensionless asymmetry parameter is

$$\tilde{\eta}_R = \frac{g_R}{2m_R^4} \eta. \quad (143)$$

Up to first order, the relations between the bare and the renormalised quantities are given by [32]:

$$\begin{aligned} m &= m_R \left\{ 1 - \frac{3}{128\pi} u_R + O(u_R^2) \right\}, \\ u &= u_R \left\{ 1 + \frac{31}{128\pi} u_R + O(u_R^2) \right\}. \end{aligned} \quad (144)$$

Expressed in terms of the renormalised parameters the nucleation rate is

$$\begin{aligned} \Gamma &= \frac{m_R^3}{u_R^{3/2} \tilde{\eta}_R^{7/3}} \exp \left[- \left(\frac{32\pi}{3} \frac{1}{u_R} + \frac{c}{2} - \frac{37}{4} + O(u_R) \right) \frac{1}{\tilde{\eta}_R^2} \right. \\ &\quad \left. + \left(\frac{8\pi}{27} (9\pi^2 - 4) \frac{1}{u_R} + 0.758 + O(u_R) \right) + O(\tilde{\eta}_R^2) \right]. \end{aligned} \quad (145)$$

This formula is an analytical expression for Γ which for the first time includes a complete treatment of quadratic fluctuations. Compared to the thin wall approximation,

$$\Gamma_{\text{TWA}} = \exp \left[- \frac{32\pi}{3} \frac{1}{u_R \tilde{\eta}_R^2} \right], \quad (146)$$

the leading term for small asymmetries η ,

$$\Gamma = \frac{m_R^3}{u_R^{3/2} \tilde{\eta}_R^{7/3}} \exp \left[- \left(\frac{32\pi}{3} \frac{1}{u_R} + \frac{c}{2} - \frac{37}{4} \right) \frac{1}{\tilde{\eta}_R^2} \right], \quad (147)$$

completes the thin wall approximation by giving the prefactor in addition to the energy of the critical bubble.

The next-to-leading terms go beyond the thin wall approximation. They can be employed to obtain an estimate for the region of validity of the small $\tilde{\eta}_R$ expansion. Comparing the terms proportional to u_R^{-1} in the exponential and requiring the correction to be smaller than the leading term we get

$$\tilde{\eta}_R < 0.65. \quad (148)$$

The result for Γ can be used to obtain estimates for the nucleation rate by inserting phenomenological or measured values for the physical parameters. Moreover it can be employed in the context of field theory for an estimate of the decay of a false vacuum at high temperatures.

Baacke and Kiselev [16] have calculated the nucleation rate in three-dimensional scalar field theory by evaluating the fluctuation determinant numerically. Our results can, however, not be compared directly, because the authors of [16] pick out the terms relevant in a certain high temperature limit. Also the renormalization scheme is different from ours.

Another numerical calculation of the nucleation rate in the framework of renormalisation group improved ef-

fective average actions has been presented in [33]. In this case, too, it is not possible without further information to compare their results with ours, because the results of [33] are expressed in terms of parameters, whose relation to the renormalised parameters used here is not clear.

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